Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 3, pp. 491–495, 1995

0021–8928/95 \$24.00+0.00

0021–8928(95)00057-7

THE AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR A NON-UNIFORM PLATE OF VARIABLE THICKNESS†

N. K. AKHMEDOV and M. F. MEKHTIYEV

Baku

(*Received* 14 *January* 1994)

The asymptotic behaviour of the axisymmetric stress–strain state of a non-uniform plate, whose thickness $h = \varepsilon r$, where r is the distance from the centre of the plate and ε is a small parameter, is investigated.

The problem of the theory of elasticity for a non-uniform hollow cone was investigated in [1], and a special case when the aperture angle of the median surface of the cone equal to $\pi/2$, which corresponds to a plate of variable thickness, was mentioned. In this paper we investigate a special form of a conical shell when its median surface degenerates to a plane. Since this case of degeneration is a special one, all the previous discussion in [1] has to be repeated.

1. Consider an elastic body in a spherical system of coordinates with the following ranges of variation of the parameters

$$r_1 \leqslant r \leqslant r_2, \quad \pi/2 - \varepsilon \leqslant \theta \leqslant \pi/2 + \varepsilon, \quad 0 \leqslant \varphi \leqslant 2\pi$$

In the axisymmetric case the equations of equilibrium have the form

$$(L_{10} + \varepsilon\partial_1 L_{11} + \varepsilon^2\partial_1^2 L_{12})\mathbf{u} = 0 \tag{1.1}$$

$$\partial[G(\partial u_\varphi + \varepsilon u_\varphi \operatorname{tg}\varepsilon\eta)] - 2G\varepsilon(\partial u_\varphi + \varepsilon u_\varphi \operatorname{tg}\varepsilon\eta)\operatorname{tg}\varepsilon\eta + \varepsilon^2 G\Delta_0 u_\varphi = 0 \tag{1.2}$$

where $\mathbf{u} = (u_r, u_\theta)^T$; u_r, u_θ, u_φ are the components of the displacement vector, L_{1k} are matrix differential operators of the form

$$L_{10} = \begin{Vmatrix} \partial G\partial - \varepsilon G \operatorname{tg}\varepsilon\eta\partial - 2\kappa\varepsilon^2 & (G+\kappa)\varepsilon^2 \operatorname{tg}\varepsilon\eta - \varepsilon(\partial G + \kappa\partial) \\ \varepsilon\partial(\kappa+\lambda) + 2\varepsilon G\partial & \partial\kappa\partial - (2G\partial + \partial\lambda)\varepsilon \operatorname{tg}\varepsilon\eta - \kappa\varepsilon^2 \sec^2 \varepsilon\eta \end{Vmatrix}$$

$$L_{11} = \begin{Vmatrix} 2\varepsilon\kappa & \lambda\partial + \partial G - \varepsilon(G+\lambda)\operatorname{tg}\varepsilon\eta \\ G\partial + \partial\lambda & 2\varepsilon G \end{Vmatrix}, \quad L_{12} = \begin{Vmatrix} \kappa & 0 \\ 0 & G \end{Vmatrix}$$

$$\Delta_0 = \partial_1^2 + 2\partial_1 - 2, \quad \partial = \frac{\partial}{\partial\eta}, \quad \partial_1^k = \rho^k \frac{\partial^k}{\partial\rho^k}, \quad \kappa = 2G + \lambda$$

$\eta = (\theta - \pi/2)\varepsilon$, $\rho = r/r_0$ are new dimensionless variables, $\varepsilon = (\theta_2 - \theta_1)/2$ is a small parameter characterizing the thickness of the plate, and $r_0 = (r_1 r_2)^{1/2}$, $\eta \in [-1, 1]$.

We will assume that the Lamé elasticity parameters $G = G(\eta)$, $\lambda = \lambda(\eta)$ are arbitrary positive piecewise-continuous functions of the variable η.

Suppose the following boundary conditions are given on the conical boundaries

$$\overline{\sigma}|_{\eta=\pm1} = M\overline{\mathbf{u}}|_{\eta=\pm1} = \overline{\mathbf{m}}^\pm(\rho) \tag{1.3}$$

$$\sigma_{\theta\varphi}|_{\eta=\pm1} = G(\varepsilon\rho)^{-1}(\partial u_\varphi + \varepsilon \operatorname{tg}\varepsilon\eta u_\varphi)|_{\eta=\pm1} = q^\pm(\rho) \tag{1.4}$$

Here

$$\overline{\sigma} = (\sigma_{r\theta}, \sigma_{\theta\theta}), \quad \overline{\mathbf{m}}^\pm = (h^\pm(\rho), f^\pm(\rho))$$

†*Prikl. Mat. Mekh.* Vol. 59, No. 3, pp. 518–523, 1995.

$$M = (\varepsilon\rho)^{-1}(M_0 + \varepsilon\partial_1 M_1)$$

$$M_0 = \begin{Vmatrix} G\partial & -\varepsilon G \\ (\kappa+\lambda)\varepsilon & \kappa\partial - \varepsilon\lambda \operatorname{tg} \varepsilon\eta \end{Vmatrix} \qquad M_1 = \begin{Vmatrix} 0 & G \\ \lambda & 0 \end{Vmatrix}$$

We will assume that the loads $h^{\pm}(\rho), f^{\pm}(\rho), q^{\pm}(\rho)$ are fairly smooth functions.
Relations (1.2) and (1.4) describe the problem of the torsion of a plate. It will be considered later.

2. To construct non-homogeneous solutions of the equation of equilibrium (1.1) which satisfy the non-homogeneous boundary conditions on the conical surfaces of the plate we can use the methods proposed for a uniform plate in [2, 3]. However, this is not the only method for removing a load from conical surfaces. To construct non-homogeneous solutions we will use the first iterative process of the asymptotic method [4].

The solution of problem (1.1), (1.3) will be sought in the form

$$u_r = \varepsilon^{-2}(u_{r0} + \varepsilon u_{r1} + \ldots), \quad u_\theta = \varepsilon^{-3}(u_{\theta 0} + \varepsilon u_{\theta 1} + \ldots) \tag{2.1}$$

Substituting (2.1) into (1.1) and (1.3) we obtain a system, the successive integration of which over η gives relations for the coefficients of the expansion of u_r and u_θ

$$\begin{aligned} u_{\theta 0} &= \varphi_1(\rho), \quad u_{r0} = \eta[\varphi_1(\rho) - \rho\varphi_1'(\rho)] + \varphi_2(\rho) \\ u_{\theta 1} &= \varphi_3(\rho), \quad u_{r1} = \eta[\varphi_3(\rho) - \rho\varphi_3'(\rho)] + \varphi_4(\rho) \end{aligned} \tag{2.2}$$

where $\mathbf{y}_1 = (\varphi_1, \varphi_2)$ and $\mathbf{y}_2 = (\varphi_3, \varphi_4)$ are the solutions of the first and second of the following equations, respectively

$$B\bar{\mathbf{y}}_1 = \bar{\mathbf{I}}_1, \quad B\bar{\mathbf{y}}_2 = \bar{\mathbf{I}}_2 \tag{2.3}$$

Here

$$B = \partial_1^4 B_4 + \partial_1^3 B_3 + \partial_1^2 B_2 + \partial_1 B_1 + B_0$$

$$B_4 = \begin{Vmatrix} 0 & 0 \\ g_2 - g_1 & 0 \end{Vmatrix}, \quad B_3 = \begin{Vmatrix} g_1 & 0 \\ 8(g_2 - g_1) & g_0 - g_1 \end{Vmatrix}$$

$$B_2 = \begin{Vmatrix} 3g_1 & -g_0 \\ 2(G_1 - G_2) + 12(g_2 - g_1) - t_1 + 2t_2 & 6(g_0 - g_1) \end{Vmatrix}$$

$$B_1 = \begin{Vmatrix} t_1 - 2G_1 & -2g_0 \\ 6(G_1 - G_2) + 3(t_2 - t_1) + g_2 & 6(g_0 - g_1) + 2(G_1 - G_0) - t_1 \end{Vmatrix}$$

$$B_0 = \begin{Vmatrix} 0 & 2G_0 \\ 0 & 4(G_1 - G_0) - g_1 \end{Vmatrix}$$

$$\bar{\mathbf{I}}_1 = (0; \rho f(\rho)), \quad \bar{\mathbf{I}}_2 = (\rho h(\rho); 4\rho h^-(\rho) + 2\rho(\rho h^-(\rho))'), \quad f(\rho) = f^+(\rho) - f^-(\rho),$$

$$h(\rho) = h^+(\rho) - h^-(\rho), \quad g_k = \int_{-1}^{1} 4G\kappa^{-1}(G+\lambda)\eta^k d\eta, \quad G_k = \int_{-1}^{1} G\eta^k d\eta, \quad t_k = \int_{-1}^{1} 2G\lambda\kappa^{-1}\eta^k d\eta$$

Note that (2.3) is a system of Euler-type equations.

3. We will construct homogeneous solutions. Put $\mathbf{m}^{\pm} = \mathbf{0}$ in (1.3). By seeking solutions of problem (1.1), (1.3) in the form

$$\bar{\mathbf{u}}(\rho, \eta) = \rho^{z - 1/2}\bar{\mathbf{w}}(\eta), \quad \bar{\mathbf{w}}(\eta) = (a, b)$$

we obtain the following non-self-conjugate eigenvalue problem

$$(L_{10} + \varepsilon(z - 1/2)(L_{11} - \varepsilon L_{12}) + \varepsilon^2(z - 1/2)^2 L_{12})\mathbf{w} = \mathbf{0} \tag{3.1}$$

$$(M_0 + \varepsilon(z - 1/2)M_1)\mathbf{w} = \mathbf{0} \quad \text{for} \quad \eta = \pm 1$$

The homogeneous solutions corresponding to the first iterative process can be obtained from relations (2.1)–(2.3) if we put $\mathbf{m}^{\pm} = \mathbf{0}$ in them. We obtain

$$u_r^{(0)} = \varepsilon C\rho^{-1}(2\eta + (2G_0)^{-1}(g_1 - 4G_1) + O(\varepsilon^2)) \tag{3.2}$$

$$u_\theta^{(0)} = C\rho^{-1}\left[1 + \varepsilon^2\left(\int_0^\eta \lambda\kappa^{-1}x\,dx + (2G_0)^{-1}(4G_1 - g_1)\eta - \eta^2\right) + O(\varepsilon^3)\right]$$

$$u_r^{(1)} = \varepsilon B(\eta - \varepsilon^2\eta^3/6 + O(\varepsilon^3)),\quad u_\theta^{(1)} = B(1 - \varepsilon^2\eta^2/2 + O(\varepsilon^3)) \tag{3.3}$$

$$u_r^{(2)} = \rho^{-1/2}\sum_{k=1}^{4} A_k u_{rk}^{(2)},\quad u_\theta^{(2)} = \varepsilon^{-1}\rho^{-1/2}\sum_{k=1}^{4} A_k u_{\theta k}^{(2)} \tag{3.4}$$

Here

$$u_{rk}^{(2)} = \{\eta(z_{k0} - 3/2)[2G_1 + (z_{k0} + 1/2)t_1 + (z_{k0} + 3/2)(2(G_0 - G_1) + (z_{k0}^2 - 1/4)(g_1 - g_0))] -$$
$$-(z_{k0} + 3/2)[(z_{k0} - 1/2)(2G_1 - 2G_2 - t_1 + t_2) - (z_{k0}^2 - 1/4)(z_{k0} - 3/2)(g_1 - g_2)] -$$
$$-2(z_{k0} - 1/2)G_2 - (z_{k0} - 1/2)^2 t_2 + O(\varepsilon^2)\}\exp(z_{k0}\ln\rho)$$
$$u_{\theta k}^{(2)} = \{(z_{k0} + 3/2)[(z_{k0}^2 - 1/4)(g_0 - g_1) + 2(G_1 - G_0)] - (z_{k0} + 1/2)t_1 - 2G_1 + O(\varepsilon^2)\}\exp(z_{k0}\ln\rho)$$

z_{k0} satisfies the biquadratic equation

$$16mz_{k0}^4 + 8(2n - m)z_{k0}^2 + (16q + m - 4n) = 0$$
$$m = g_1^2 - g_0g_2,\quad n = 2G_0g_2 + 6g_0G_2 - 8g_1G_1$$
$$q = 16G_1^2 - 12G_0G_2 - 4G_1g_1$$

and C, B and A_k are unknown constants.

Solutions (3.2) and (3.3) correspond to the eigenvalues $z_0^{(0)} = -1/2$ and $z_0^{(1)} = 1/2$.

We will now consider the following iterative process.

The solution of problem (3.1) will be sought in the form

$$a^{(3)}(\eta) = \varepsilon(a_{30} + \varepsilon a_{31} + \ldots),\quad b^{(3)}(\eta) = \varepsilon(b_{30} + \varepsilon b_{31} + \ldots) \tag{3.5}$$
$$z_k = \varepsilon^{-1}(\beta_{k0} + \varepsilon^2\beta_{k2} + \ldots)$$

Substituting (3.5) into (3.1) we obtain, after reduction

$$u_r^{(3)} = \rho^{-1/2}\varepsilon\sum_{k=1}^{\infty} F_k u_{rk}^{(3)},\quad u_\theta^{(3)} = \rho^{-1/2}\varepsilon\sum_{k=1}^{\infty} F_k u_{\theta k}^{(3)}$$
$$u_{rk}^{(3)} = [p_0\beta_{k0}^{-2}\psi_k'' - p_2\psi_k + O(\varepsilon)]\exp(\varepsilon^{-1}\beta_{k0}\ln\rho)$$
$$u_{\theta k}^{(3)} = [-\beta_{k0}^{-3}(p_0\psi_k'')' - 2\beta_{k0}^{-1}\psi_k' + \beta_{k0}^{-1}(p_2\psi_k)' + O(\varepsilon)]\exp(\varepsilon^{-1}\beta_{k0}\ln\rho) \tag{3.6}$$
$$p_0 = \kappa(4G(G + \lambda))^{-1},\quad p_1 = (2G)^{-1},\quad p_2 = \lambda(4G(G + \lambda))^{-1}$$

Here $\psi_k(\eta)$ are the solutions of the generalized Papkovich eigenvalue problem for the non-homogeneous case [5, 6].

We will now analyse the stress–strain state corresponding to the different groups of solutions.

The stress state defined by solution (3.2) is equivalent to the principal force vector P directed along the axis of symmetry. The constant C is then related to P by the following relation

$$P = 2\pi r_0^2\varepsilon^3 C[6G_2 + 2G_1G_0^{-1}(g_1 - 4G_1) + O(\varepsilon)] \tag{3.7}$$

The principal stress vector in the section $r = \text{const}$ for the remaining homogeneous solutions is zero.

The stress state defined by solution (3.3) corresponds to the displacement of the plate as a solid.

The stress state defined by (3.4) is equivalent to forces T_r and T_φ and bending moments M_r and M_φ, referred to the median plane of the plate.

It can be seen from (3.6) that the first terms of the asymptotic expansions of the stresses and strains are identical with the boundary-layer type solutions for a non-uniform plate of constant thickness [5, 6].

4. We will consider the problem of the removal of stresses from the side surface of the plate. Suppose we are given the following stresses for $\rho = \rho_s$ $(s = 1, 2)$

$$\sigma_{rr} = \sigma_s(\eta),\quad \sigma_{r\theta} = \tau_s(\eta) \tag{4.1}$$

The functions $\sigma_s(\eta)$, $\tau_s(\eta)$ are fairly smooth and satisfy the conditions of equilibrium.

As was noted above, the non-self-balanced part of the stresses can be removed using the penetrating solution (3.2), where the relation between the constant C and the principal force vector P is given by (3.7). We will therefore assume below that $P = 0$. Then, using Lagrange's variational principle, we obtain the following infinite system of algebraic equations

$$\sum_{k=1}^{\infty} m_{jk}C_k = N_j \quad (j=1,2,\ldots) \tag{4.2}$$

$$m_{jk} = \sum_{s=1}^{2} \exp(z_j + z_k)\ln\rho_s \int_{-1}^{1} (Q_{rk}u_{rj} + T_k u_{\theta j})\cos\varepsilon\eta d\eta$$

$$N_j = \sum_{s=1}^{2} \exp(z_j + 3/2)\ln\rho_s \int_{-1}^{1} (\sigma_s u_{rj} + \tau_s u_{\theta j})\cos\varepsilon\eta d\eta$$

(The notation is the same as in [1].)

Using the fact that the parameter ε characterizing the thickness of the plate is small, we can construct asymptotic solutions of (4.2). The matrices of these systems are known from the theory of non-uniform plates of constant thickness [5, 6] and will therefore not be given here. Numerical versions of the different problems were investigated using them. The conditions for this system to be solvable are considered in [7].

5. We will now investigate the torsion problem. The solution of problem (1.2), (1.4) will be sought in the form

$$u_\varphi = \varepsilon^{-1}(u_{\varphi 0} + \varepsilon u_{\varphi 1} + \varepsilon^2 u_{\varphi 2} + \ldots) \tag{5.1}$$

Substituting (5.1) into (1.2) and (1.4) we obtain relations for the coefficients of the expansion of u_φ

$$u_{\varphi 0} = g_0(\rho), \quad u_{\varphi 1} = g_1(\rho) \tag{5.2}$$

$$u_{\varphi 2} = -\frac{1}{2}\eta^2 g_0(\rho) + \frac{\rho q(\rho)}{G_0}\int_0^\eta G^{-1}\int_{-1}^{y} G dx dy + \rho q^-(\rho)\int_0^\eta G^{-1}dx + g_2(\rho)$$

$$\Delta_0 g_0 = -\frac{\rho q(\rho)}{G_0}, \quad \Delta_0 g_1 = 0, \quad \Delta_0 g_2 = \frac{(2G_0 - 3G_2)}{2G_0^2}\rho q(\rho) -$$

$$-\frac{\Delta_0(\rho q(\rho))}{G_0^2}\int_{-1}^{1} G\int_0^\eta G^{-1}\int_{-1}^{y} G dx dy d\eta - \frac{\Delta_0^*(\rho q^-(\rho))}{G_0}\int_{-1}^{1} G\int_0^\eta G^{-1}dx d\eta$$

$$q(\rho) = q^+(\rho) - q^-(\rho)$$

We will now consider the problem of constructing homogeneous solutions. Put $q^\pm(\rho) = 0$ in (1.4). After separating the variables, by representing the solution in the form

$$u_\varphi(\rho,\eta) = \rho^{z-1/2}\upsilon(\eta)$$

we obtain the following self-conjugate spectral problem for the function $\upsilon(\eta)$

$$L\upsilon = (9/4 - z^2)\upsilon$$

$$L\upsilon = \{\varepsilon^{-2}G^{-1}\partial[G(\partial\upsilon + \varepsilon\upsilon\,\mathrm{tg}\,\varepsilon\eta)] - 2\varepsilon^{-1}(\partial\upsilon + \varepsilon\upsilon\,\mathrm{tg}\,\varepsilon\eta)\mathrm{tg}\,\varepsilon\eta; \quad G(\partial\upsilon + \varepsilon\upsilon\,\mathrm{tg}\,\varepsilon\eta)|_{\eta=\pm 1} = 0\} \tag{5.3}$$

It can be proved that the operator L is self-conjugate in Hilbert space $L_2(-1, 1)$ with weight $G(\eta)\cos\varepsilon\eta$.

Putting $q^\pm(\rho) = 0$ in (5.1) and (5.2) we obtain homogeneous solutions corresponding to the first iterative process

$$u_\varphi^{(0)} = D\rho^{-2}\cos\varepsilon\eta \tag{5.4}$$

where D is an unknown constant.

The eigenvalue $z_0 = -3/2$ corresponds to this solution.

Using the next iterative process we will seek the solution of problem (5.3) in the form

$$\upsilon_k = \upsilon_{k0} + \varepsilon^2\upsilon_{k2} + \ldots, \quad z_k = \varepsilon^{-1}(\delta_{k0} + \varepsilon^2\delta_{k2} + \ldots) \tag{5.5}$$

Substituting (5.5) into (5.3) we obtain

$$\upsilon_{k2} = \sum_{p=0}^{\infty} \alpha_{kp}\upsilon_{p0}, \quad \alpha_{kp} = \frac{1}{(\delta_{k0}^2 - \delta_{p0}^2)}\int_{-1}^{1} [2G\upsilon_{k0}' - (G\upsilon_{k0})']\upsilon_{p0}\eta d\eta, \quad \alpha_{kk} = \frac{1}{4}\int_{-1}^{1} G\eta^2\upsilon_{k0}^2 d\eta$$

$$\delta_{k2} = \frac{2}{2\delta_{k0}}\left(\frac{5}{4} + \int_{-1}^{1} [2G\upsilon_{k0}' - (G\upsilon_{k0})']\upsilon_{k0}\eta d\eta\right)$$

Here υ_{k0} are the solutions of the torsional problem for a non-uniform plate of constant thickness [5, 6]

$$-G^{-1}(G\upsilon_{k0}')' = \delta_{k0}^2\upsilon_{k0}, \quad G\upsilon_{k0}'|_{\pm 1} = 0 \tag{5.6}$$

Solution (5.4) defines the internal stress–strain state of the plate. The constant D is proportional to the torsional moments of the stresses acting in the section r = const

$$M_t = -6\pi\varepsilon r_0^3 D \int_{-1}^{1} G\cos^3 \varepsilon\eta d\eta$$

The stress state corresponding to the second group of solutions has the form of a boundary layer.

Suppose we are given the following stresses on the side surface of the plate

$$\sigma_{r\varphi} = Gf_s(\eta) \quad \text{for} \quad \rho = \rho_s \quad (s = 1, 2) \tag{5.7}$$

The non-self-balanced part of the stresses can be removed using the penetrating solution (5.4). We will assume that $M_t = 0$. By virtue of this assumption we have $D = 0$.

We will represent the variables in the form

$$u_\varphi = \sum_{k=1}^{\infty} (D_k\rho^{-(z_k+1/2)} + E_k\rho^{z_k-1/2})\upsilon_k(\eta)\sqrt{G\cos\varepsilon\eta} \tag{5.8}$$

Using (5.8) we obtain

$$\sigma_{r\varphi} = \sum_{k=1}^{\infty} G[E_k(z_k - 3/2)\rho^{z_k-3/2} - D_k(z_k + 3/2)\rho^{-(z_k+3/2)}]\upsilon_k(\eta)\sqrt{G\cos\varepsilon\eta} \tag{5.9}$$

To satisfy boundary conditions (5.7) we will expand the specified functions $f_s(\eta)$ ($s = 1, 2$) in series in eigenfunctions of the eigenvalue problem (5.3)

$$f_s(\eta) = \sum_{k=1}^{\infty} a_{sk}\upsilon_k(\eta)\sqrt{G\cos\varepsilon\eta} \tag{5.10}$$

$$a_{sk} = (f_s, \upsilon_k) = \int_{-1}^{1} Gf_s(\eta)\upsilon_k(\eta)\cos\varepsilon\eta d\eta$$

$$(\upsilon_k, \upsilon_n) = \delta_{kn}, \quad \|\upsilon_k\|^2 = 1 = \int_{-1}^{1} G\upsilon_k^2(\eta)\cos\varepsilon\eta d\eta$$

Substituting (5.9) and (5.10) into (5.7) we obtain the following expression which enables us to find the constants E_k and D_k

$$[(z_k - 3/2)\rho^{z_k-3/2}E_k - (z_k + 3/2)\rho^{-(z_k+3/2)}D_k]|_{\rho=\rho_s} = a_{sk}$$

All the solutions obtained above are identical with the solutions for a uniform plate [2, 3] when G = const, λ = const.

REFERENCES

1. AKHMEDOV N. K. and MEKHTIYEV M. F., Analysis of the three-dimensional problem of the theory of elasticity for a non-uniform hollow frustum of a cone. *Prikl. Mat. Mekh.* **57**, 5, 113–119, 1993.
2. MEKHTIYEV M. F., The construction of more accurate applied theories for a plate of variable thickness. *Izv. Akad. Nauk AzSSR. Ser. Fiz.-Tekh. i Mat. Nauk* 3, 47–52, 1979.
3. MEKHTIYEV M. F. and USTINOV Yu. A., The asymptotic behaviour of the solution of the problem of the theory of elasticity for a plate of variable thickness. In *Proceedings of the 8th All-Union Conference on the Theory of Shells and Plates*, Rostov-on-Don, 1971. Nauka, Moscow, 58–61, 1973.
4. GOL'DENVEIZER A. L., The construction of an approximate theory of shells using asymptotic integration of the equation of the theory of elasticity. *Prikl. Mat. Mekh.* **27**, 4, 593–608, 1963.
5. VOROVICH I. I., KADOLITSEV I. G. and USTINOV Yu. A., The theory of plates of non-uniform thickness. *Izv. Akad. Nauk SSSR. MTT* 3, 119–129, 1975.
6. USTINOV Yu. A., Some properties of the homogeneous solutions of non-uniform plates. *Dokl. Akad. Nauk SSSR* **216**, 4, 755–758, 1974.
7. USTINOV Yu. A. and YUDOVICH V. I., The completeness of the system of elementary solutions of the biharmonic equation in a half-strip. *Prikl. Mat. Mekh.* **37**, 4, 706–714, 1973.

Translated by R.C.G.